Factuality and Backward Induction
with Arbitrary Choice Functions

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Durham University

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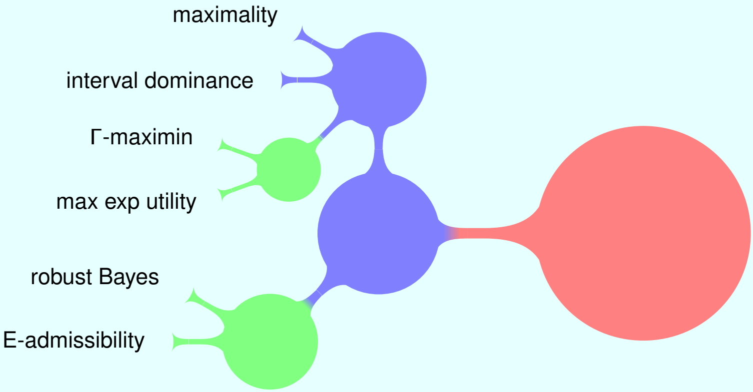
Outline

Outline

Problem Description



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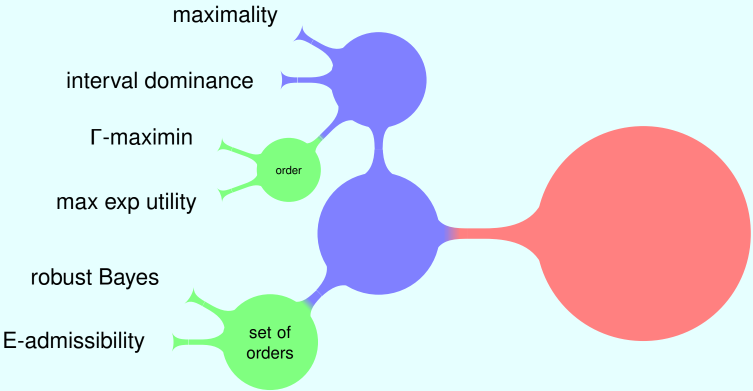

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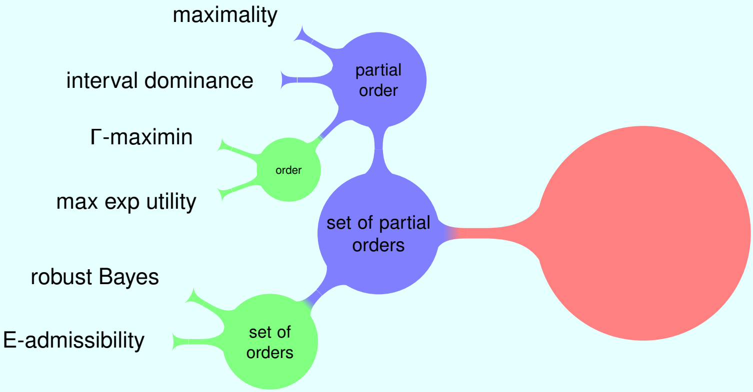

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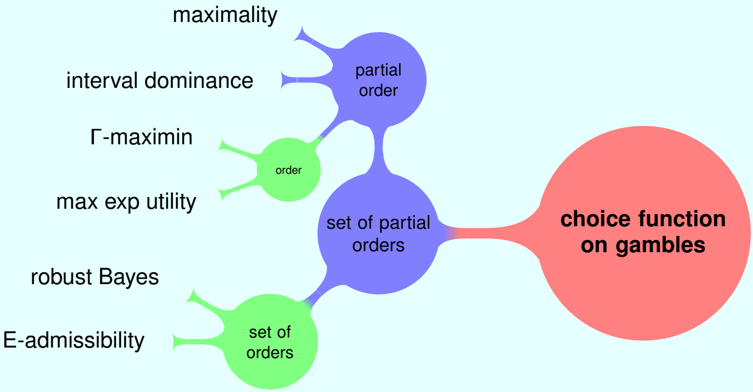

Gambles and Choice Functions

Definition

A gamble is an uncertain reward, i.e. a mapping from the possibility space Ω to the reward set $\mathcal{R}$.

"probabilityless (horse-)lottery"

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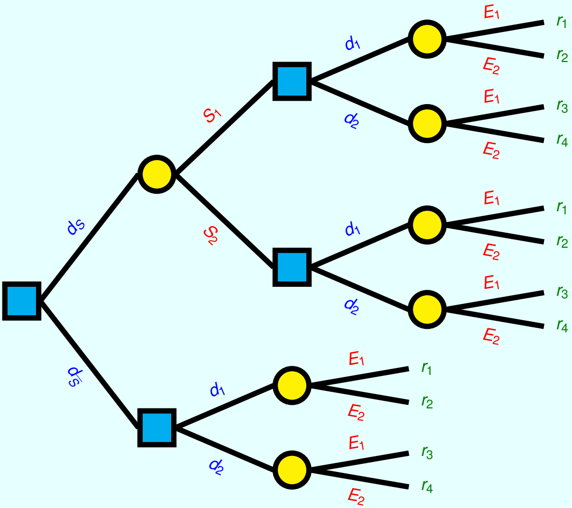
How to solve sequential decision problems
with a choice function?

Decision Trees: Example



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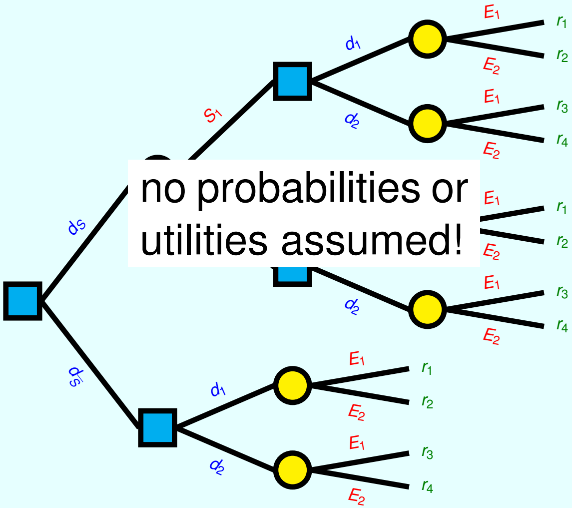
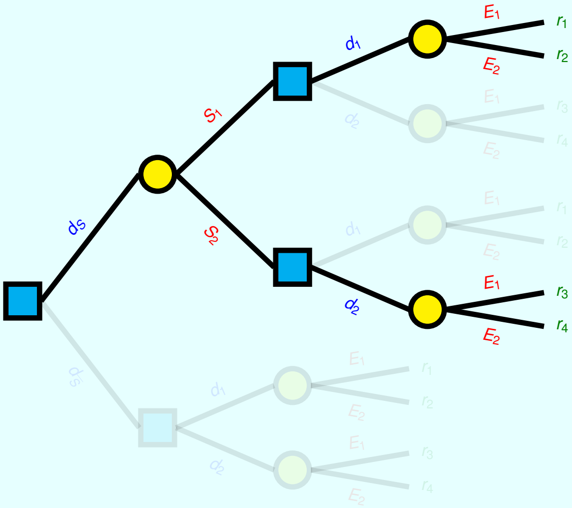


no probabilities or utilities assumed!

Decision Trees: Normal Form Decisions



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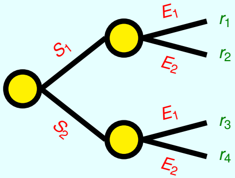

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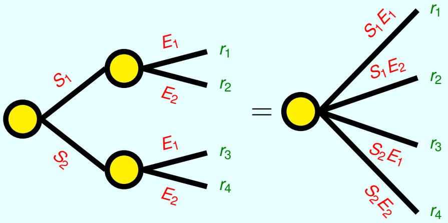

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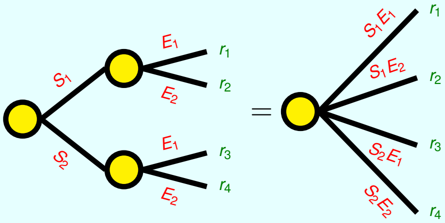

$$= X(\omega) = \begin{cases} r_1 & \text{if } \omega \in S_1 E_1 \\ r_2 & \text{if } \omega \in S_1 E_2 \\ r_3 & \text{if } \omega \in S_2 E_1 \\ r_4 & \text{if } \omega \in S_2 E_2 \end{cases}$$

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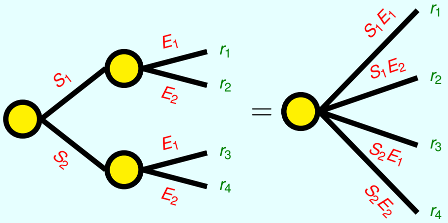

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Observation

Every normal form decision induces a gamble.

Decision Trees: Normal Form Solution

Definition

A normal form solution of a decision tree is a set of these normal form decisions.

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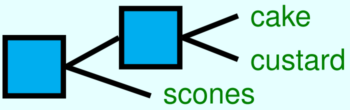
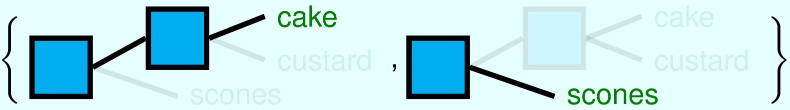
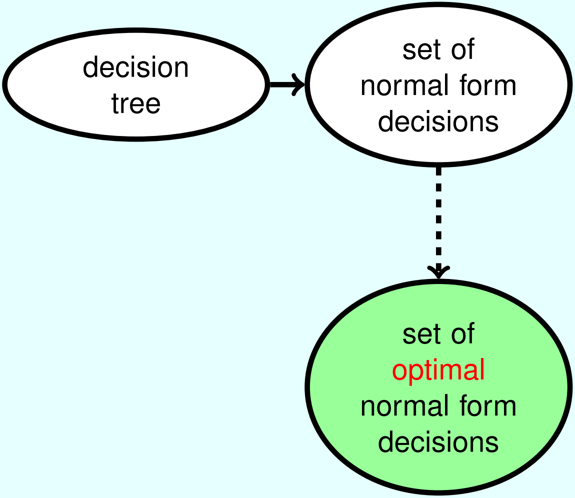
for example



could have as normal form solution



Decision Trees: Normal Form Solution Induced By opt



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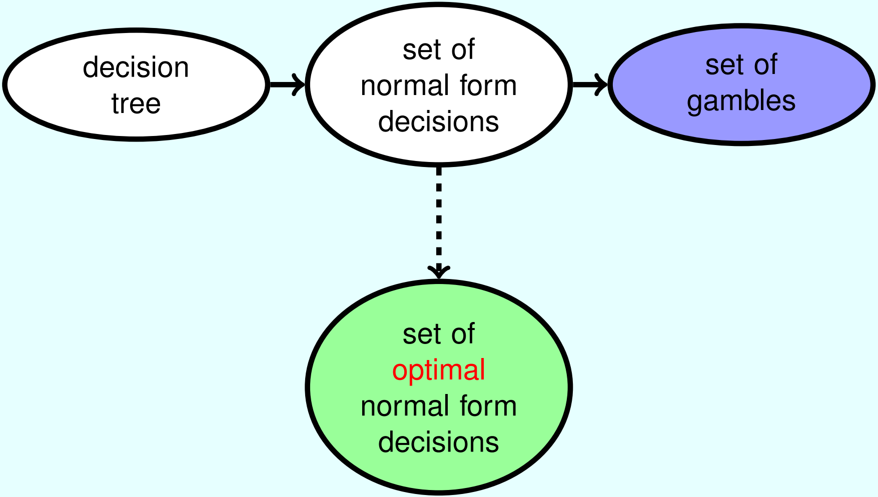

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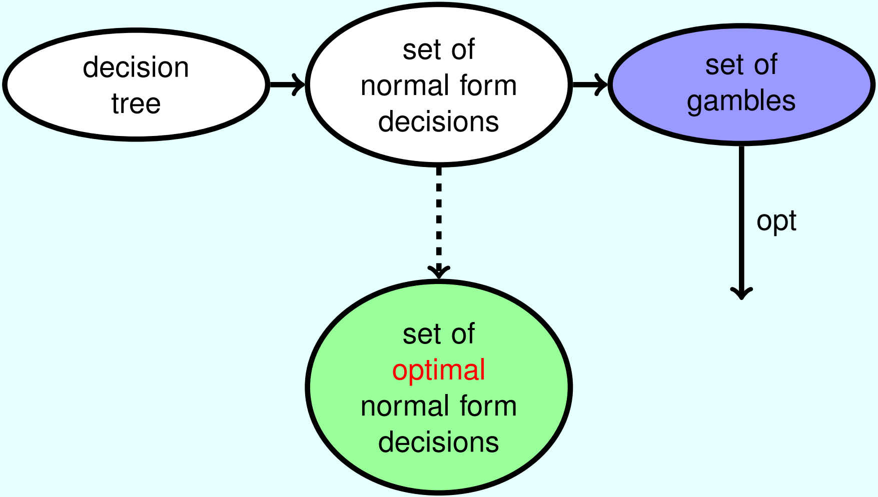

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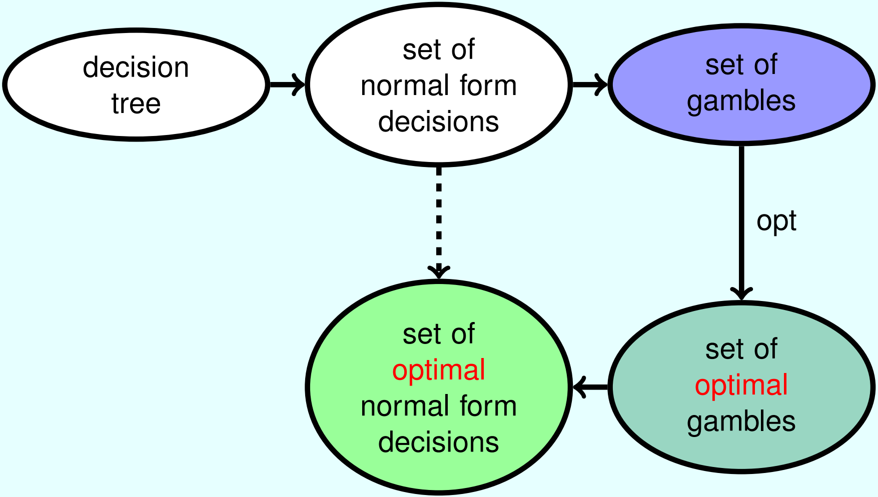

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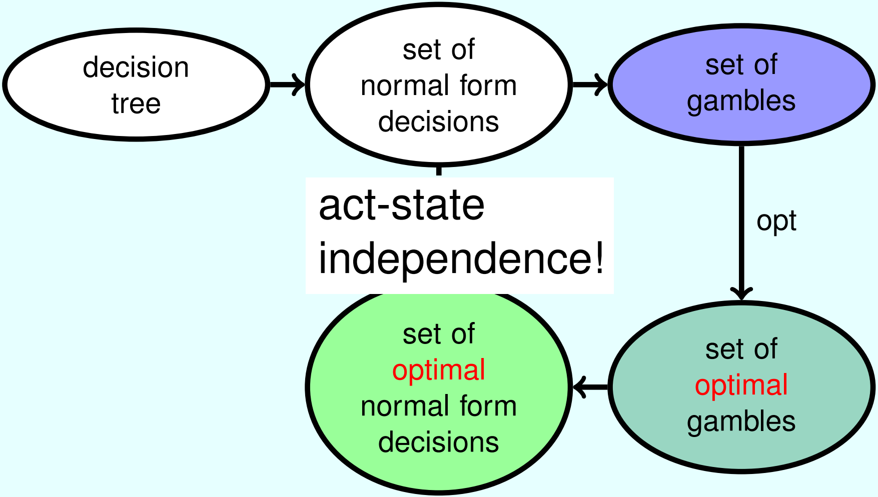
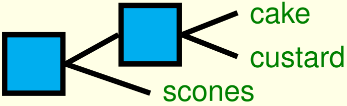


Outline

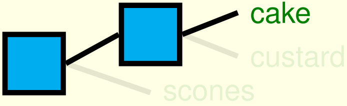
Factuality: A Counterfactual Example

Suppose opt (



)

= {



}

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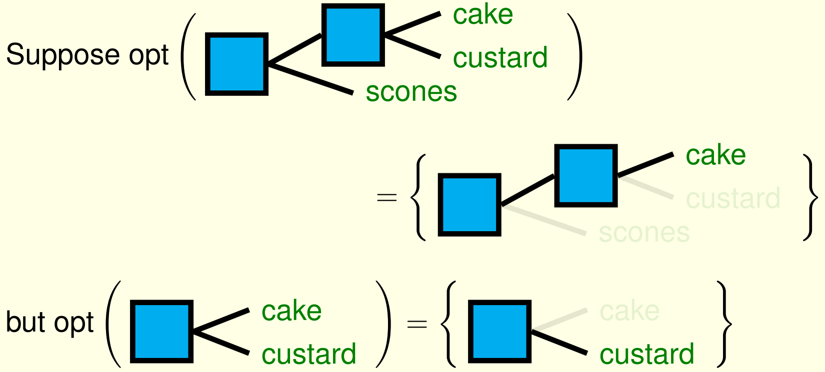

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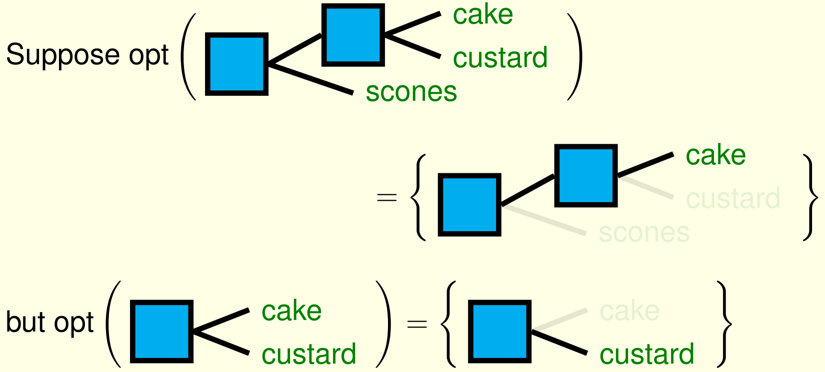


The choice between cake and custard
depends on the tree in which the decision is embedded.

Factuality: Definition

Definition

opt is called factual whenever for every decision tree

$$\text{restriction}(\text{opt}(\text{tree})) = \text{opt}(\text{restriction}(\text{tree}))$$

whenever restriction(tree)'s root node is in opt(tree).

In bargaining theory this principle is called subgame perfection.

Factuality Theorem

Theorem

opt is factual if and only if it satisfies:

- Conditioning property. If $\{X, Y\} \subseteq \mathcal{X}$ and $AX = AY$, then
$$X \in \text{opt}(\mathcal{X}|A) \iff Y \in \text{opt}(\mathcal{X}|A).$$
- Intersection property. If $\mathcal{Y} \subseteq \mathcal{X}$ and $\text{opt}(\mathcal{X}|A) \cap \mathcal{Y} \neq \emptyset$, then
$$\text{opt}(\mathcal{Y}|A) = \text{opt}(\mathcal{X}|A) \cap \mathcal{Y}.$$
- Mixture property.
$$\text{opt}(A\mathcal{X} \oplus \overline{A}Z|B) = A\,\text{opt}(\mathcal{X}|A \cap B) \oplus \overline{A}Z.$$

Note: some technical details omitted.

Factuality: No Imprecision

Total Preorder Theorem

The intersection property is equivalent to:

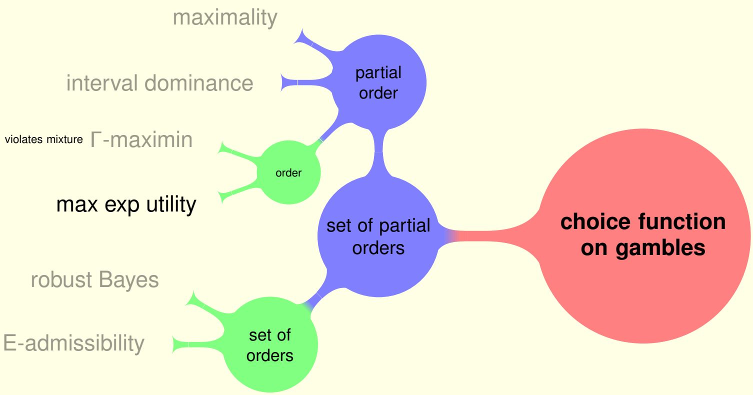
- Total preorder property. For every event $A \neq \emptyset$, there is a total preorder $\succeq_A$ on gambles such that

$$\mathrm{opt}(\mathcal{X}|A) = \max_{\succeq_A}(\mathcal{X})$$

So it is impossible to be at the same time

- factual, and
- optimal with respect a non-total preference ordering (such as for instance a partial preference ordering)

Factuality: What Choice Functions are Factual?



Factuality: What Can Be Done?

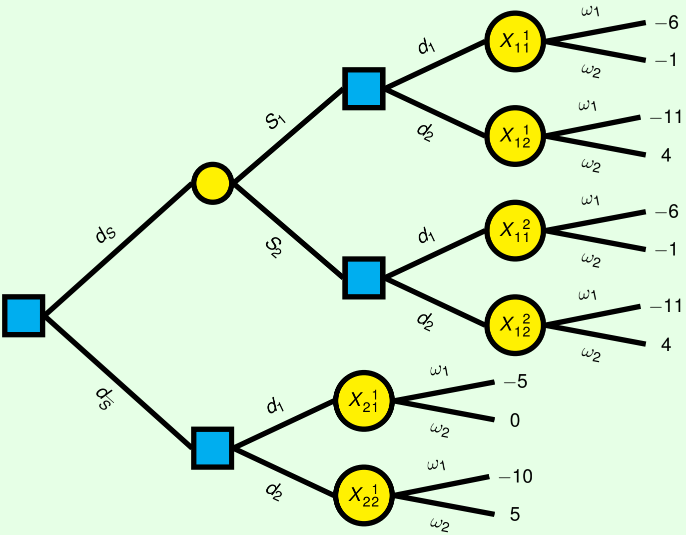
- Some types of counterfactuality may not be so bad, for instance those where backward induction still works (such as maximality and E-admissibility).
- Restrict type of decision trees that you are interested in: there are sequential decision processes where factuality can be obtained under substantially weaker assumptions.

Outline

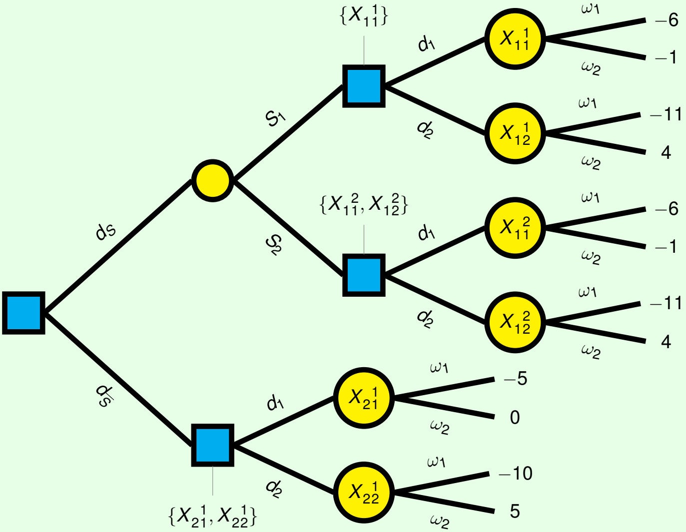
Backward Induction

- Idea of backward induction: use the solutions of subtrees to eliminate many options in the full tree
- For weak orders, there is a natural method of backward induction
- For choice functions corresponding to partial orders (maximality, interval dominance) or more complicated choice functions (E-admissibility) there are several possibilities
- The goal of our method is to find the normal form solution induced by opt

Example


d_S
$d_{\bar{S}}$
S_1
S_2
d_1
d_2
X_{11}^1
X_{12}^1
X_{11}^2
X_{12}^2
X_{21}^1
X_{22}^1
ω_1
ω_2
-6
-1
-11
4
-6
-1
-11
4
-5
0
-10
5

Example


$\{X_{11}^1\}$
$\{X_{11}^2, X_{12}^2\}$
$\{X_{21}^1, X_{22}^1\}$
d_S
$\overline{d_S}$
S_1
S_2
d_1
d_2
d_1
d_2
d_1
d_2
X_{11}^1
X_{12}^1
X_{11}^2
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-10
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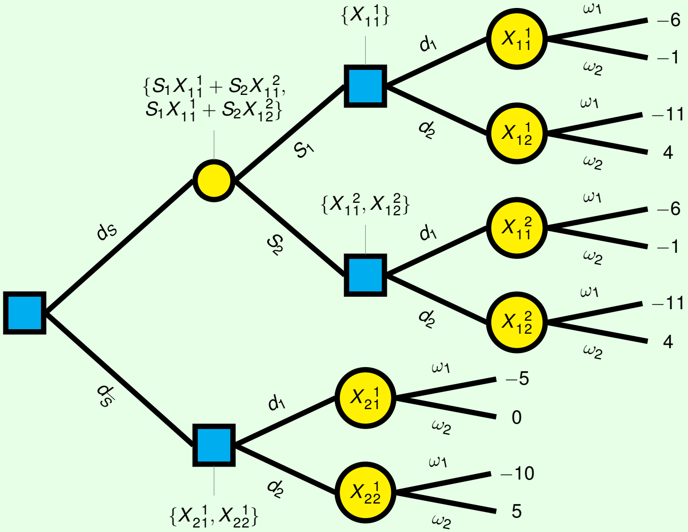
Example


$\{X_{11}^1\}$
$\{S_1X_{11}^1 + S_2X_{11}^2,$
$S_1X_{11}^1 + S_2X_{12}^2\}$
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$\{X_{21}^1, X_{22}^1\}$
d_S
$d_{\bar{S}}$
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S_2
d_1
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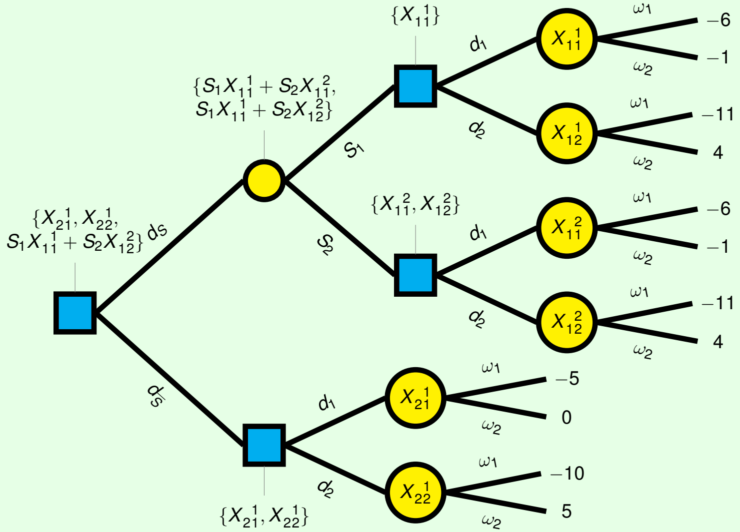
Example


$\{X_{11}^1\}$
$\{S_1X_{11}^1 + S_2X_{11}^2, S_1X_{11}^1 + S_2X_{12}^2\}$
$\{X_{11}^2, X_{12}^2\}$
$\{X_{21}^1, X_{22}^1, S_1X_{11}^1 + S_2X_{12}^2\}$
$\{X_{21}^1, X_{22}^1\}$
d_S
S_1
S_2
d_1
d_2
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−6
−1
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4
−6
−1
−11
4
−5
0
−10
5

When Can Backward Induction Be Used?

- When opt is factual, backward induction will work
- When opt is particularly counterfactual, such as in the cake/custard/scones example, it will not work

When Can Backward Induction Be Used?

- When opt is factual, backward induction will work
- When opt is particularly counterfactual, such as in the cake/custard/scones example, it will not work
- What about when the local solution is a superset of the global solution?

Necessary and Sufficient Conditions

Theorem

Backward induction works with opt if and only if it satisfies:

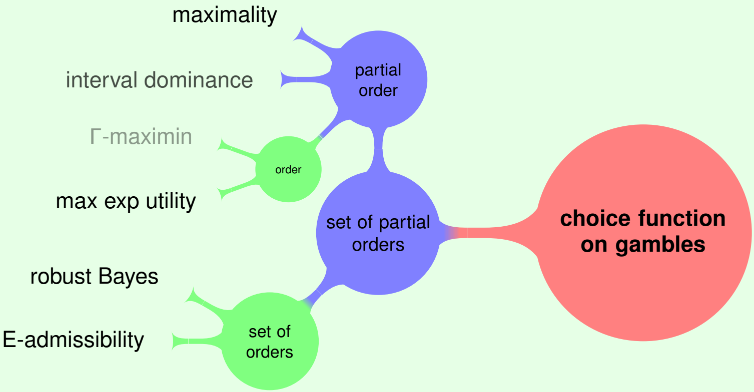
- Backward conditioning property. If $AX = AY$ and $\{X, Y\} \subseteq \mathcal{X}$, then $X \in \mathrm{opt}(\mathcal{X}|A) \iff Y \in \mathrm{opt}(\mathcal{X}|A)$ (subject to some technicalities)
- Path independence.
$$\mathrm{opt}\left(\bigcup_{i=1}^{n} \mathcal{X}_i \middle| A\right) = \mathrm{opt}\left(\bigcup_{i=1}^{n} \mathrm{opt}(\mathcal{X}_i|A) \middle| A\right)$$
- Backward mixture property.
$$\mathrm{opt}\left(\{AX + \overline{A}Z \colon X \in \mathcal{X}\}|B\right) \subseteq A\,\mathrm{opt}(\mathcal{X}|A \cap B) \oplus \overline{A}Z$$

Note: some technical details omitted.

When Can Backward Induction Be Used?

- If opt is factual then indeed backward induction works
- The opposite implication is not true
- If opt satisfies the conditions, then the local solutions are supersets of the global solution (or they are equal)
- The opposite implication is not true (path independence may still fail)
- If the local solutions are subsets of the global solution, then backward induction may be useful to find a subset of the global solution

Backward Induction: What Choice Functions Work?



Conclusion

- Conditions on opt can be thought of as rationality constraints on choice
- Factuality is sufficient but not necessary for backward induction
- If opt is counterfactual but backward induction works, then knowing counterfactual information refines one's optimal decisions
- In this case, no extreme differences between global and local solutions (unlike many counterfactual weak orders)